Q: Find the arc length of the parametric curve defined by $x(t) = \cos t + \ln\left(\tan\frac{t}{2}\right)$ and $y(t) = \sin t$ on the interval from $\frac{\pi}{4} \le t \le \frac{3\pi}{4}$.

A: First, compute the derivatives :

$$\frac{\mathrm{d}x}{\mathrm{d}t} = -\sin t + \frac{\sec^2\frac{t}{2}}{2\tan\frac{t}{2}} = -\sin t + \frac{1}{2}\sec\frac{t}{2}\csc\frac{t}{2}$$

$$\frac{\mathrm{d}y}{\mathrm{d}t} = \cos t$$

$$s = \int_{\pi/4}^{3\pi/4} \mathrm{d}t\sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)}$$ definition of arc length parameterized by *t*

$$= \int_{\pi/4}^{3\pi/4} \mathrm{d}t\sqrt{\left(-\sin t + \frac{1}{2}\sec\frac{t}{2}\csc\frac{t}{2}\right)^2 + \sin^2 t}$$ substitute derivatives

$$= \int_{\pi/4}^{3\pi/4} \mathrm{d}t\sqrt{\cos^2 t - \sin t\sec\frac{t}{2}\csc\frac{t}{2} + \frac{1}{4}\sec^2\frac{t}{2}\csc^2\frac{t}{2} + \sin^2 t}$$ expand square

$$= \int_{\pi/4}^{3\pi/4} \mathrm{d}t\sqrt{1 - \sin t\sec\frac{t}{2}\csc\frac{t}{2} + \frac{1}{4}\sec^2\frac{t}{2}\csc^2\frac{t}{2}}$$ Pythagorean identity

$$= \int_{\pi/4}^{3\pi/4} \mathrm{d}t\sqrt{\csc^2 t - 1}$$ half-angle identities

$$= \int_{\pi/4}^{3\pi/4} |\cot t|\,\mathrm{d}t$$ Pythagorean identity

$$= 2\int_{\pi/4}^{\pi/2} \cot t\mathrm{d}t$$ cot *t* is negative over part of the interval and symmetric around $\frac{\pi}{2}$

$$= 2\ln(\sin t)\Big|_{\pi/4}^{\pi/2}$$ integrate

$$= 2\left(\ln 1 - \ln\frac{\sqrt{2}}{2}\right)$$ subsitute limits

logarithmic identities

$$= 2\ln\sqrt{2}$$

$$= \boxed{\ln 2}$$ logarithmic identity